Lesson 4: The Present Value of an Annuity

Here we will learn about a very important formula: the present value of an annuity. This formula is used whenever there is a series of identical payments—whether going towards paying off a debt, supporting someone (including yourself) during retirement, paying off a house, compensating someone for injury, or any other reason. Before we begin to use this very powerful formula, we will derive two ways. The first represents a mathematician’s view, and the second an economist’s view. Hopefully, one view or the other will match your learning style. Alternatively, you can just memorize the formula, so long as you know how to use it.

How many times have we heard a television commercial say “12 easy equal monthly payments” for ownership of some snazzy product? An annuity is actually this sequence of equally spaced payments of equal amounts. This payment method applies to an enormous number of financial situations. Our goal in the next few boxes is to derive and explain the following:

$$PV = c \cdot \frac{1 - (1 + i)^{-n}}{i}$$

which will be used to solve a whole range of problems, from car loans to home mortgages and all sorts of other financial arrangements.

Do you remember the formula for a geometric series that “stops early”? If not, then go reread it on Page 455, before reading the rest of this box. Let us say that Chuck has a mortgage, whose interest compounds monthly at 6%. He is sending in checks at the value of $2000 on the first of the month, every month for 30 years. On the day that this arrangement is contracted, how much is this sequence of payments worth, from the bank’s perspective? This will be the mortgage amount, in dollars, that Fred should expect to receive from the bank upon signing the contract. Let us assume that the contract is signed one month before the first payment, as is standard practice.

In this case then the first payment occurs after one month, the second after two months, as well as the third after three months. The nth payment occurs after n months. Using the Time Value of Money formula (see Page 429), we would then value the first payment at $2000/(1 + i)^1$, the second payment at $2000/(1 + i)^2$, and so forth. This gives us:

$$PV = \frac{2000}{(1 + i)^1} + \frac{2000}{(1 + i)^2} + \frac{2000}{(1 + i)^3} + \cdots + \frac{2000}{(1 + i)^{359}} + \frac{2000}{(1 + i)^{360}}$$

where 360 = 30 × 12 is the number of payments (more precisely, the total number of months in 30 years).

Yet, what is i? Since r = 0.06 and m = 12, then $i = 0.06/12 = 0.005$. Substituting that we get:

$$PV = \frac{2000}{(1.005)^1} + \frac{2000}{(1.005)^2} + \frac{2000}{(1.005)^3} + \cdots + \frac{2000}{(1.005)^{359}} + \frac{2000}{(1.005)^{360}}$$

Does this series look familiar? It should, because we calculated it on Page 455. Its value is $333,583, and is how much the bank will give Fred for this mortgage agreement.

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Now suppose I have a three-year car loan with monthly payments. My monthly payments will last for 3 years, which comes to $3 \times 12 = 36$ payments. The first payment will be one month after purchase, followed by the second payment two months after, then three months after, and so on until thirty six months have passed. Using the Time Value of Money formula (see Page 429) I can compute the value of each payment.

As with the previous example, the $n$th payment occurs after $n$ months. If $i$ is the prevailing rate of interest (written monthly) and $c$ is the dollar value of the check I send the bank each month, then the present value of the $n$th payment is

$$PV = \frac{FV}{(1 + i)^n} = \frac{c}{(1 + i)^n}$$

The total value of all 36 payments would be their sum:

$$PV = c \frac{1}{(1 + i)^1} + c \frac{1}{(1 + i)^2} + c \frac{1}{(1 + i)^3} + \cdots + c \frac{1}{(1 + i)^{36}}$$

At this point please pause and take a moment to convince yourself that this is a geometric series, stopping early, with first entry $c/(1 + i)$ and last term $c/(1 + i)^{36}$, and common ratio $1/(1 + i)$. Only once you understand how this series is composed should you continue to the next box, where we will calculate the value.

Okay, so the sum of the above geometric series (stopping early) is:

$$S = a - c r z$$

In the previous box, we computed that the first entry is $a = c/(1 + i)$. We also computed that the last entry is $z = c/(1 + i)^{36}$ and that the common ratio is $c r = 1/(1 + i)$. We should plus these in now, and we obtain

$$S = \frac{c}{1+i} - \frac{\frac{1}{1+i} \cdot \frac{c}{(1+i)^{36}}}{1 - \frac{1}{1+i}}$$

Now we will multiply the numerator and denominator by $(1 + i)$ and obtain:

$$S = c \cdot \frac{1}{1 + i} - \frac{c}{1 + i - 1}$$

$$S = c \cdot \frac{1 - (1 + i)^{-36}}{i}$$
A sequence of payments equally spaced throughout time and of equal value is called an *annuity*. In the previous box we derived the formula for an annuity with 36 payments, but keep in mind that any number of payments is acceptable. If there are \( m \) payments per year for \( t \) years, then \( n = mt \) is the number of total payments. The value of each payment is \( c \). If the interest rate is \( r \), then let \( i = r/m \) as usual. The present value of this sequence of payments, then, is calculated by

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i}
\]

and this is a formula that we’re going to get some serious mileage out of.

We just saw the classic mathematician’s derivation of the present value formula for an annuity. The payments (after adjustment by the Time Value of Money Formula) form a geometric series that stops early, so we naturally use the formula for geometric series stopping early to find the sum. There’s really nothing else to it.

The economist’s derivation of the present value of an annuity is a bit different, but brings you to the same answer. It can be described as a “series of observations.” We will begin with those observations in the next box.

We will now approach the derivation of the PV formula from the perspective of an economist or financier. Before we begin you should probably review the concept of a perpetuity-forborne, given on Page 448.

- Let’s say (very hypothetically) that you have a perpetuity-due with the bank, and that the bank has a perpetuity-due with you. That means that you give the bank a check for \( c \) dollars on the first of every month, and they also present you with a check for \( c \) dollars on the first of every month. Remember, with a perpetuity, the payments never end. In this case, the whole thing is a complete wash. The payments cancel out and nothing happens financially.

- Now let’s say that the bank “forbears” its perpetuity by three years. This means you will have a proper perpetuity-due—giving the bank a check for the first three years on the first of the month, every month, until the end of time—but the bank delays by 37 months and you get a check from them starting only on the month after the three year anniversary—month 37. This will be followed by a check from them in month 38, month 39, month 40, and so on.

- The previous bullet can be thought of in two stages: Stage One is the first 36 months, and Stage Two is everything afterward. In Stage Two the situation is identical to the first bullet in this list, so financially nothing is happening. In Stage One, however, you’re giving the bank a check on the first of the month, month after month, for three years.

- Stage One represents a standard loan with \( 3 \times 12 = 36 \) equal monthly payments. It is a regular schedule of identical and equally spaced payments, and therefore an annuity.

- What is the present value of this arrangement? The value of the perpetuity-due that you are paying the bank with your monthly checks (all of value \( c \)) would be \( PV = c/i \). The value of the perpetuity-forborne (with \( n = 37 \)) that you are getting from the bank is

\[
PV = \frac{c}{i(1 + i)^{37} - 1} = \frac{c}{i(1 + i)^{36}}
\]

which is based on the formula from Page 448.

Now we can complete this derivation in the next box.
Continuing with the previous box, we know that the entire arrangement’s value must be the perpetuity-due minus the perpetuity forborne. The calculation isn’t so bad:

\[
\frac{c}{i} - \frac{c}{i(1+i)^{36}} = \frac{c(1+i)^{36} - c}{i(1+i)^{36}} - \frac{c}{i(1+i)^{36}}
\]

\[
= \frac{c(1+i)^{36} - c}{i(1+i)^{36}}
\]

\[
= \frac{c}{i}[1 - \frac{1}{(1+i)^{36}}]
\]

\[
= \frac{c}{i} \frac{1 - (1 + i)^{-36}}{1 - (1 + i)^{-36}}
\]

\[
= \frac{c}{i}
\]

We’ve derived this important formula now, with two very different points of view. I hope that you understood at least one or the other derivation. If you understood both, then that’s phenomenally good news. However, if you understood neither, you have two choices. You can just memorize the formula if you like—but that’s accident prone and also won’t assist you in using the formula correctly. Alternatively, you could reread one of the derivations.

Now that we have derived our formula, let’s use it!

Perhaps I can afford a car payment of $300 a month. If I want a five-year car loan, what should the value of my loan be? Let the interest rate be 4.80%. We begin with the formula

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 300 \cdot \frac{1 - (1 + 0.004)^{-5\times12}}{0.004} = 300 \cdot \frac{0.212995\cdots}{0.004} = 15,974.66
\]

where \(i = 0.004\) because \(i = r/m = 0.048/12\). So I should get a car loan for $15,974.66.

What if the car loan from the previous example is not enough? What payment \(c\), all other factors being the same, is required in order to get a $20,000 loan?

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 20,000 \cdot \frac{1 - (1 + 0.004)^{-5\times12}}{0.004} = 20,000 \cdot \frac{0.212995\cdots}{0.004} = 20,000 \cdot c(375.59 \cdots) = 375.59 = c
\]

Well that wasn’t so bad, but it would be nice if there were a shortcut!
What if I wanted a $40,000 car? Or a $10,000 car? What is the monthly payment in each case?

[Answer: $751.19 and $187.79.]

Note, that’s exactly double and half the previous example, respectively.

These computations are not horribly long, but they are probably too much for the average bank employee to perform on the spot. Instead, we need a shortcut.

As you saw in the previous boxes, if you double the PV then you double the payment $c$; if you halve the PV then you halve the payment $c$. We can conclude that the PV and payment are directly proportional. That enables us to use the “cost per thousand” technique for annuities. (We first saw that technique on Page 260.)

First, use $PV = 1000$ to find $c$. Then if someone wants a $40,000 car, just multiply the “cost per thousand” by 40 to find their payment. Likewise, if someone wants a $30,000 car, multiply the “cost per thousand” by 30 to find their payment. If you have many calculations to do, this can be a huge time-saver.

Suppose you work for a car dealership that is offering 2.5% financing compounded monthly, with no down-payment. What is the cost-per-thousand for a 3-year car loan?

First we must find $i$. That is $0.025/12 = 0.002083$. Then we use

$$PV = c \cdot \frac{1 - (1 + i)^{-36}}{i}$$

$$1000 = c \cdot \frac{1 - (1 + 0.002083)^{-36}}{0.002083}$$

$$1000 = c \cdot \frac{0.0721841\cdots}{0.002083}$$

$$1000 = c \cdot 34.6483\cdots$$

$$\frac{1000}{34.6483\cdots} = 28.8613\cdots = c$$

We learn that the cost-per-thousand is $28.8613\cdots$. This would be one of those times where you do not round money to the nearest penny, because you’d be destroying a good deal of accuracy if you did that.

These problems can be a bit longer than what you’re used to. The trick to doing them successfully is to allow your pencil to share the burden with your brain. First, write out the formula in full. Then it is much easier to plug in the correct values. Next, proceed to slowly move toward the correct answer by calculation and algebraic manipulation. Not only does this reduce the chances of an error being made, but also it increases the partial credit you can obtain. You’ve shown the instructor that you know both the formula and also where to plug in the data.
The large fraction \( \frac{1 - (1 + i)^{-n}}{i} \)

keeps coming up in our calculations. This is sometimes called the *growth factor*, but less commonly the *compounding factor* or the *power of compounding*. (We’ll use the term “growth factor.”) You might want to review the concept of “compounding factor” from Page 260. This is an important value, and it reflects the impact of the payment \( c \). In the previous example, the fraction evaluated to 34.6483 \( \cdots \) and multiplying this by the monthly payment of \$28.86 yields

\[
28.8613 \cdots \times 34.6483 \cdots = 999.99
\]

with the missing penny just being rounding-error.

It turns out that finance people have their own symbol for the growth factor because it is so important. We will have no use of the symbol, but you should see it once so that if you see it later in another class or during a job interview you will not panic. Finance people will often write

\[
a_{\overline{n}} = \frac{1 - (1+i)^{-n}}{i}
\]

to represent the growth factor of a series of payments.

If a loan is to be monthly for five years, and the interest rate is 7%, then…

- …what is the cost per thousand? [Answer: $19.8011 \cdots .]
- …what is the compounding factor? [Answer: 50.5019 \cdots .]

Suppose my friend loans me some money, which I want to pay back in five months. The loan is for \$2000, and we agree upon an interest rate of 10% (compounded monthly) because he’s selling investments to make the loan to me.

- How much should each payment be? [Answer: \$410.05.]
- What is the compounding factor? [Answer: 4.87739 \cdots .]
- What is the cost-per-thousand? [Answer: \$205.027 \cdots .]
An “Amortization Table” for a loan can be an excellent way to understand what is going on inside the loan. It is very easy to make one. In the first column of the table I will write the heading “Period #” and then “Starting Value” in the second. In the third column I will put “Interest Accrued” and in the fourth column I will put “Payment.” Finally, the fifth and last column is “Ending Balance.”

We will use the loan from the previous problem. The starting principal is $2000, so for the first period that goes under “Starting Value.” The interest is going to be \( i \) times the starting value for that period, which is 2000 \( \times \) 0.1/12 = 16.67. (Notice I’m deviating from our normal rounding policy of truncation. For this problem we’ll round to the nearest numbers, because in an amortization table rounding errors accrue very fast and build on each other.) The payment is always $410.05, as we calculated in the previous box. The ending balance for Period 1 is therefore 2000 + 16.67 – 410.05 = 1606.62.

For the next month, the starting value is the ending balance of the previous month, or $1606.62. Then the interest rate is 1606.62 \( \times \) 0.1/12 = 13.39. The payment remains $410.05, and we have the ending balance as 1606.62 – 13.39 + 410.05 = 1209.96.

The remaining three months work just like the second month. The table is given in the next box.

The amortization table from the previous example would be tabulated for Row 3, Row 4, and Row 5 by the same methods as Row 2. After that, the complete table should look like:

<table>
<thead>
<tr>
<th>Period #</th>
<th>Starting Value</th>
<th>Interest Accrued</th>
<th>Payment</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2000.00</td>
<td>$16.67</td>
<td>$410.05</td>
<td>$1606.62</td>
</tr>
<tr>
<td>2</td>
<td>$1606.62</td>
<td>$13.39</td>
<td>$410.05</td>
<td>$1209.96</td>
</tr>
<tr>
<td>3</td>
<td>$1209.96</td>
<td>$10.08</td>
<td>$410.05</td>
<td>$809.99</td>
</tr>
<tr>
<td>4</td>
<td>$809.99</td>
<td>$6.75</td>
<td>$410.05</td>
<td>$406.69</td>
</tr>
<tr>
<td>5</td>
<td>$406.69</td>
<td>$3.39</td>
<td>$410.05</td>
<td>$0.03</td>
</tr>
</tbody>
</table>

(The extra three cents at the end is rounding error that we’ll address two boxes from now.)

Now try it yourself with a $3000 loan, with interest 12% compounded monthly, and duration of six months. Use the PV formula to find the payment, which should be $517.64. Once you have that, construct the Amortization Table.

The solution is given at the end of the lesson, on Page 489.

The extra three cents two boxes ago might bother some students. Let’s explore it this way:

Consider the $2000 loan at 10% compounded monthly, for which we just did an amortization table. What will 5 months at 10% compounded monthly produce...

- . . . for a payment of $410.06? [Answer: $2000.02.]
- . . . for a payment of $410.05? [Answer: $1999.97.]
This rounding discrepancy almost always happens. The math actually calls for a payment of
\[ c = 410.05532370105295 \ldots \]
but there’s no way to pay for
\[ \$ 0.00532370105295 \ldots \]
so mathematical correctness just is not possible. You can either round up to 410.06 or down to 410.05. In practice, banks will round upward to the nearest penny, thus you slightly overpay them, when you take out a loan. If they rounded downward, then you would slightly underpay them. It is important to realize that we are only talking about a few cents for small loans, and even for home mortgages it is never as large as a $10 difference.

Consider a mortgage of $500,000, at 7% interest for 30 years. Of course, it is compounded monthly, as are all mortgages.

- What value of \( c \) does the math call for? [Answer: 3326.5124758959211 \ldots .]
- What is the PV of an annuity at 7% compounded monthly for 30 years and a payment of $3326.51? [Answer: $499,999.62.]
- What is the PV of an annuity at 7% compounded monthly for 30 years and a payment of $3326.52? [Answer: $500,001.13.]

As you can see from the previous box, the fact that we cannot pay the exact value of \( c \) is a matter of $1.51 even for a half-million dollar house. This tiny difference is not worth fretting over or spending more time on.

The purpose of this box and the previous two boxes was to teach you that...

- rounding error does exist, and it must be taken into account, but...
- rounding \( c \) upward to the nearest penny will ensure that this is not a problem for any business.

Suppose I can afford a monthly house payment of $2000. The going rate for mortgages is perhaps 7%. I have $35,000 in my savings account for the down payment, which my bank tells me must be 10% of the value of the house (or more). How much of a house can I afford? I would like a 30-year mortgage.

First, we should find \( i \) and \( n \). We have \( i = r/m = 0.07/12 = 0.005833 \), and \( n = m \times t = 12 \times 30 = 360 \). Now we can use our formula:

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} \\
PV = 2000 \cdot \frac{1 - (1 + 0.005833)^{-360}}{0.005833} \\
PV = 2000 \cdot \frac{0.876794 \ldots}{0.005833} \\
PV = 300,615.13 
\]

Thus I learn that I’ll get $300,615.13 from my mortgage. The maximum value I can purchase would therefore be 300,615.13 + 35,000 = 335,615.13.

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Let’s reconsider the previous example, if the interest rate were instead only 5%.

- How much is the value of a mortgage with 5% interest, $2000 monthly payments, and 30 years duration? [Answer: $372,563.23.]

- Alternatively, if there is a house costing $335,615 which I’d like, and I use $35,000 for the down payment, and allocate $300,615 for the mortgage, what will my monthly payment be? [Answer: $1613.77.]

Now we can see why lowering the interest rate can be a boon to the economy. When the Federal Reserve Bank (“The Fed”) lowers the prime rate, then banks can give lower interest rates on mortgages. Note, each individual bank still decides what rate to charge. However, the interest rates are interconnected in ways that we should probably not go into now.

In the previous checkerboard box, we saw that when the interest rate fell from 7% to 5%, the future homebuyer could get a mortgage for $372,563 instead of for only $300,615. That’s a huge difference! It means that people can afford larger homes, or even to build new homes, and that will employ people in the construction industry. Also, when people move into larger homes, state, county, and town governments can collect more in property taxes, causing growth in schools, roads, mass transit, prisons, and bureaucratic jobs to oversee that growth.

We also saw that if the homeowner decided to stick with the $335,615 house, then instead of paying $2000 per month, the monthly payment would fall to $1613.77. That extra $386.23 per month is extremely significant. It could be spent on vacations, entertainment, or a new car. This would help the tourism, night life, or automobile industry. Likewise, the money could also be invested, and that would (slightly) raise the value of all stocks as more money flowed into the market.

With the previous box in mind, we should also consider the “dark side” of lowering the interest rate. First, imagine yourself as the seller of that $335,615 house. Imagine that you have several potential customers who are looking to spend around $320k to $340k, when the interest rate is 7%. Suddenly, the interest rate is now 5% and they all can afford around $360k to $380k! Is it not tempting to try to ask more for your house? After all, if you receive several offers, you can begin to negotiate and try to get a better price.

For this reason, housing prices can rise and during the housing boom of 2001–2005, they rose out of control to very high prices. This is called “a bubble,” and caused a whole host of problems. We don’t have time to go into that now, but you can see how interest rates can have a huge impact on the economy.

It is worthwhile to make a note that the interest rate would never change from 7% to 5% overnight. Usually the increments are 1/4th of 1%, or 25 basis-points. However, the example is easier to understand with a larger (though unrealistic) change.

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If you get a mortgage from a bank, and then cannot make the payments, then the bank loses a great deal of money. Of course, the bank can foreclose on the mortgage but then they have to seize the house, evict the residents, fix it up, put the house on the market, find a buyer, and close the sale. All that requires a lot of work by highly paid professionals.

One of the ways that the bank can avoid this fiasco is to make sure that people do not purchase more house than they can afford. Two major strategies for that would be to set a minimum downpayment size, and to ask the future homeowner what their maximum monthly payment could be, and then limit the homebuyer according to these two restrictions.

Let’s suppose that my bank requires 10% down. I can afford $2000 per month, and the bank is offering a 7% mortgage. Furthermore suppose that I can afford a downpayment of $35,000. How much of a house can I afford?

The down payment would be at least 10% of the mortgage, and is limited to $35,000; likewise the mortgage would be at most $300,615.13. Why? Because we calculated the PV of a mortgage of 30 years, 7% interest, and \( c = 2000 \) as an example on Page 475, and we learned that the present value of such a sequence of payments is $300,615.13

If \( p \) is the price of the house, then 0.1\( p \) is the minimum down payment (limited to $2000) and 0.9\( p \) is the maximum mortgage (limited to $300,615.13). So I would write

\[
0.1p \leq 35,000
\]

which becomes \( p \leq 350,000 \). Now we’ve modeled the downpayment, so we should model the PV of the mortgage as well. We would write

\[
0.9p \leq 300,615.13
\]

which becomes

\[
p \leq \frac{300,615.13}{0.9} = 334,016.81
\]

to take care of the PV of the mortgage.

The final answer is the stricter of these two requirements. Therefore, I must restrict myself to houses that cost \$334,016.81 or less, to satisfy both requirements.

Repeat the previous example if I have only $25,000 in savings. All other details remain the same. How much of a house can I afford? What requirement, the downpayment or the monthly payment, is the restriction?

[Answer: In this case, the 10% down-payment is the limitation, and I am restricted to houses that cost up to \$250,000.]

Repeat the previous example if I have $30,000 in savings, but the interest rate becomes 9% for mortgages. (That’s a bit high, but it happened during the Reagan era.) All other details remain the same. How much of a house can I afford? What requirement, the downpayment or the monthly payment, is the restriction?

[Answer: In this case, the monthly payment is the restriction. I am limited to houses that cost \$276,181.92 or less.]
Well, that was depressing. Let’s try some uplifting examples.

Let’s suppose I budget conservatively. Imagine that I buy a smaller/cheaper house than I can afford because I don’t want an oppressive mortgage payment each month. I’m going to buy a $250,000 house, paying a 10% down payment. Let’s suppose the interest rate is 6.50%.

- What is the value of the down payment? [Answer: $25,000.]
- What must the PV of the mortgage be? [Answer: $225,000.]
- What is the required monthly payment, \( c \)? [Answer: $1422.15.]

Now, in the next box, we’re going to explore how I can save some money in the long run, by overpaying the bank each month.

Continuing with our analysis from the previous box, now suppose that I pay the bank double the monthly payment every month. You’re always allowed to simply overpay. (Some mortgages—but very few—will have pre-payment penalties; because these are rare, we’ll ignore them.) Suppose you pay \( 2 \times 1422.15 = 2844.30 \) per month.

which is double the required amount. This is an enormous financial sacrifice, but as we are about to see, an extremely effective one. How long will the mortgage take to be paid? (In other words, what is the value of \( n \) when \( PV = 225,000 \) and \( c = 2844.30 \)?)

\[
225,000 = 2844.30 \cdot \frac{1 - (1 + 0.0054166)^{-n}}{0.0054166}
\]

\[
\frac{225,000 \cdot 0.0054166}{2844.30} = 1 - (1.0054166)^{-n}
\]

\[
0.428488 \cdots = 1 - (1.0054166)^{-n}
\]

\[
0.571511 \cdots = (1.0054166)^{-n}
\]

\[
\log(0.571511 \cdots) = \log \left( (1.0054166)^{-n} \right)
\]

\[
-0.242975 \cdots = -n \log(1.0054166)
\]

\[
-0.242975 \cdots = -n(0.00234608 \cdots)
\]

\[
103.566 \cdots = n
\]

Thus, after 104 months (because 103 won’t be quite enough, being less than the \( n \) that the math found) you will be mortgage-free. That’s 8 years and 8 months—far, far shorter than 30 years!

I think it is worthwhile to reflect upon the power of the financial technique in the previous example. If I get the mortgage at age 28, then I would be making the $1422.15 per month payments for 30 years, or until I am age 58. However, if I pay the $2844.30, then I will be finished at age 36 (and 8 months.) This way, I would be able to enjoy all that extra money each month, from age 37 until age 58.

Let’s look at how to quantify how much I am saving.

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The total paid to the bank for the mortgage would be the sum of all the values of \( c \). In the previous box, in the original mortgage, there were 30 \( \times 12 = 360 \) payments of $1422.15, which is a total of

\[
1422.15 \times 360 = 511,974.00
\]

In return for this, you were paid $225,000 to buy the house with. The question is, where did the $511,974.00 \( - \) $225,000 = $286,974.00 end up going? That is the total interest paid. In return for being lent $225,000, you have given the bank more than a quarter of a million dollars. This is why banks like to issue mortgages! This is how they make money.

In the second case, there were 104 payments. The last payment would be fractional, and we’ll learn how to tabulate that on Page ?? . A reasonably close model is to take the 104th payment as equal to the other 103 payments. That comes to

\[
104 \times 2844.30 = 295,807.20
\]

and the total interest paid in this case is

\[
295,807.20 - 225,000 = 70,807.20
\]

Now we can see how much we are saving, by analyzing the mathematics of the previous box. If we follow the original plan, we are paying $286,974 in interest. However, if we pay double the required payment per month, are giving paying $70,807.20 in interest.

This means that we would save

\[
286,974.00 - 70,807.20 = 216,166.80
\]

by paying double the normal payment each month.

That’s a huge quantity of money!

The mortgage from two boxes ago was realistic, but paying double the payment voluntarily is extraordinary and uncommon. Suppose, more realistically, that you pay 10% more than the required payment.

- What is the payment? [Answer: The payments are $1564.36.]
- How many terms does the mortgage take to be paid off? [Answer: \( n = 279.508 \cdots \) months which means 280 months or 23 years and 4 months.]
- What is the total interest paid? [Answer: $213,020.80.]

We will now repeat the analysis of the previous box, but supposing that you pay 20% more.

- What is the monthly payment? [Answer: $1706.58.]
- How many terms does the mortgage take to be paid off? [Answer: 231.815 \cdots \) months, which is really 232 months or 19 years and 4 months.]
- What is the total interest paid? [Answer: $170,926.56.]

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Re-examining the previous box, take a moment to reflect on how great a deal this overpayment of 10% turns out to be. Let’s compare it to the original loan. Instead of 30 years, you will make 19 years and 4 months of payments. Instead of giving $511,974 in interest to the bankers, you will be giving them $170,926.56 in interest.

The best way to check a problem of the type we just saw is to notice that we are claiming that the loan will be paid off in between the 231st and 232nd payments—if you pay $1,706.58.

Therefore, to be really precise, you can calculate the PV of an annuity with the given interest rate of 6.5% and the payment of $1,706.58. You should calculate it twice, once with \( n = 231 \) and once with \( n = 232 \). If you are in a rush on an exam, calculating only one of these will probably suffice.

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 1706.58 \cdot \frac{1 - (1 + 0.0054166)^{-231}}{0.0054166} = 1706.58 \cdot \frac{0.712885 \cdots}{0.0054166} = 224,602.29
\]

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 1706.58 \cdot \frac{1 - (1 + 0.0054166)^{-232}}{0.0054166} = 1706.58 \cdot \frac{0.714432 \cdots}{0.0054166} = 225,089.63
\]

As you can see, they straddle the correct value of $225,000, as desired.

To check the other checkerboard, where we overpaid by 10%, we just change the payment to $1,564.36 and the number of periods to \( n = 279 \) and \( n = 280 \). Then we have

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 1564.36 \cdot \frac{1 - (1 + 0.0054166)^{-279}}{0.0054166} = 1564.36 \cdot \frac{0.778464 \cdots}{0.0054166} = 224,824.33
\]

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 1564.36 \cdot \frac{1 - (1 + 0.0054166)^{-280}}{0.0054166} = 1564.36 \cdot \frac{0.779657 \cdots}{0.0054166} = 225,169.03
\]

Once again, they straddle the correct value of $225,000, as desired.

The total interest paid, as before, is the sum of all the monthly payments minus the value of the loan. Do not include the down payment.

The finance charge is the total interest paid plus any fees. However, the fees on mortgages and other loans are often extremely complex, so we will save this detail for future coursework.

Let’s suppose a payment plan for a home-entertainment system valued at $2,995 is for 12 equal monthly payments of $269.95.

- What is the total of all the payments? [Answer: $3,239.40.]
- What is the total interest paid? [Answer: $244.40.]
- Note: Since there are no fees, this is also the finance charge.
- The total of all the payments represents what percentage markup of the original price? [Answer: A markup of 8.16%.]

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In the previous box, we are making monthly payments of $269.95, which seems reasonable, and the markup is only 8.16%. Thus, we might be conned into believing that this is not a terribly bad loan. Careful analysis will now prove otherwise.

As you can see, the schedule of payments is fixed, and all the payments are equal. While 12 months is much shorter than the 30 years of a mortgage, mathematically the formulas of an annuity still apply. (This is because all the payments are the same, there is a fixed number of them, and they are equally spaced.)

I am now going to claim that the interest rate is 14.73%. You’ll learn to calculate that yourself very shortly (on Page ??), but for now you should verify my claim. We are paying monthly, so \( m = 12 \) and thus \( i = r/m = 0.1473/12 = 0.012275 \). Next, \( c = 269.95 \), and so we are ready to plug into our formula:

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 269.95 \cdot \frac{1 - (1 + 0.012275)^{-12}}{0.012275} = 269.95 \cdot \frac{0.136190 \cdots}{0.012275} = 2995.08
\]

which is close enough.

A Pause for Reflection…

In the previous example, we learned two important lessons. While the interest rate appeared on an intuitive level to be low, we see that in fact the rate is 14.73%, which is rather high but not outrageously high.

- We learned a technical finance lesson: computing the percentage markup required to change the value of the item into the sum of the payments, while mathematically valid, tells you little about the interest rate of the loan.
- However, we also saw a broader lesson that has appeared before: compound interest is counter-intuitive; do not trust your intuition. Calculation, on the other hand, yields reliable answers.

In the home-entertainment system example above, we saw how the percentage markup told us an underestimate of the interest rate. You might be wondering if this is always the case. It turns out that sometimes the percentage markup is an overestimate.

Consider the original mortgage that we analyzed on Page 478. The payments were each $1422.15, monthly for 30 years—in return for $225,000 to buy the house with. (Note: we exclude the initial down payment of $25,000.)

- What is the sum of all the payments? [Answer: $511,974.]
- What is the markup? [Answer: A markup of 127.54% would turn $225,000 into $511,974.]

As you can see, 127.54% is much larger than 6.5%, which is the actual interest rate of the loan.

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Because calculating a percentage markup is a mathematical operation that is both easy and frequently used, business students are very familiar with it. However, the percentage markup in an annuity problem does not tell you the interest rate. Sadly, many business students think that it does.

We saw that in one case, the percentage markup was a drastic underestimate of the true interest rate. We saw in the other case, that the percentage markup was a phenomenal overestimate of the true interest rate. The bottom line is that the percentage markup is not a useful tool in analyzing an annuity.

There’s a car that you want selling for $8995, and you put an initial $1000 down payment on it. You obtain a car loan for 9% compounded monthly. (This is not a great rate, actually, but it can happen if your credit rating is poor.)

- What is the monthly payment on the $7995 loan if you get a 3-year loan? [Answer: $254.23.]
- How about a 4-year loan? [Answer: $198.95.]
- And a 5-year loan? [Answer: $165.96.]

The following problem was suggested by Joel Casser, a childhood friend of mine who works at a car dealership.

We’re going to look at a car loan and make the problem as realistic as possible. You wish to purchase a car, whose total cost is $29,995 plus 7.5% sales tax. The required down payment is 20%. You’re trading a car worth $7500. Thus you need to pay

$$29,995 - 7500 = 22,495$$

to which we apply the sales tax and get

$$22,495(1.075) = 24,182.12 \cdots$$

You have three options. The first option is 0% financing for three years. The second option is 2.99% financing for 3 years, but with a $5000 cash-back incentive. The third option is a $6000 rebate, but 3.25% financing. All three options are compounded monthly. Your money-market account is currently yielding 3.50%, compounded daily. (Use a 360-day year.)

Since the amount left is $24,182.12 then a 20% down payment will be $24,182.12 \times 0.2 = 4836.42$ and the loan itself will have a principal of $24,182.12 - 4836.42 = 19,345.70.$

Now we’ll investigate all three options.

The 0% financing case is easy. This is probably the first time you’ve seen 0% financing in this book, so don’t feel bad if you didn’t know what to do. We’ll examine this situation a bit more on Page 521. As you can see, zero-percent financing is just a long way of saying that the payments are an equal division of the cost of the item, with no interest paid at all. The shortcut for calculating this is to realize that there are 36 equal monthly payments, so the payments will be $19,345.70/36 = 537.38.$
The 2.99% case is just a present-value calculation. We know that the rate \( r = 0.0299 \) is compounded monthly, so \( m = 12 \) and then we have \( \frac{i}{m} = 0.0299/12 = 0.0024916666 \). The present value is the amount to be paid minus the cash-back incentive, or

\[
19,345.70 - 5000 = 14,345.70
\]

Since the loan is for 3 years, there will be \( n = 36 \) payments. Then we have

\[
14,345.70 = c \cdot \frac{1 - (1 + i)^{-n}}{i} = c \cdot \frac{1 - (1 + 0.0024916666)^{-36}}{0.0024916666} = c \cdot \frac{0.0856925 \cdots}{0.0024916666} = (34.3916 \cdots)c
\]

We then can calculate \( c = 14,345.70/34.3916 = 417.12 \cdots \). That is very significantly cheaper!

The third option is more interesting. We have \( r = 3.25\% \) therefore \( i = 0.0325/12 = 0.0027083 \). The present value is $19,345.70. Then we have

\[
19,345.70 = c \cdot \frac{1 - (1 + i)^{-n}}{i} = c \cdot \frac{1 - (1 + 0.0027083)^{-36}}{0.0027083} = c \cdot \frac{0.0927781 \cdots}{0.0027083} = (34.2565 \cdots)c
\]

and then we can calculate \( c = 19,345.70/34.2565 = 564.73 \cdots \). That’s the most expensive payment yet. However, that doesn’t mean that it is the worst option.

Wait! What about the $6000 rebate? This is money that is mailed to you after the fact—you simply get a check for this amount. Perhaps you’ll use it to replenish your checking account after making the down payment, or perhaps you’ll place it in a savings account, in case you’re in danger of missing a few payments. Alternatively, you might have some other plans for the money. If you have no use for it, then you should take into consideration that you could invest it and earn interest. In any case, this is a human consideration that mathematical finance cannot address.

Finally, the real way to settle the issue is to see how much is paid for the car in each case.

- In the first case, I have payments of $537.38 for 36 months or a total of $537.38 \times 36 = 19,345.68.
- In the second case, I have payments of $537.38 for 36 months or a total of $537.38 \times 36 = 15,016.32, much better.
- In the third case, I have payments of $564.73 for 36 months or a total of $564.73 \times 36 = 20,330.28, which looks high, but once we subtract the $6000 rebate we get $14,330.28. This turns out to actually be the best option.

Suppose in a state with 5% sales tax, you are trading in a $5000 car for an $18,500 car. You can have 6% financing with a $4000 cash-back incentive, or you can have 0% financing. The loan is for three years and is compounded monthly.

- If you choose 0% financing, what is the value of your loan? [Answer: $14,175.]
- If you choose 6% financing, what is the value of your loan? [Answer: $10,175.]
- If you choose 0% financing, what is the monthly payment? [Answer: $393.75.]
- If you choose 6% financing, what is the monthly payment? [Answer: $309.54.]
- What is the total that you pay for the car in the 0% case? [Answer: $14,175.]
- What is the total that you pay for the car in the 6% case? [Answer: $11,143.55.]
- Which is the better deal? [Answer: the cash-back incentive, clearly.]
While problems involving the present value of an annuity formula are usually about debts, they also can go in the reverse direction. The key is that there must be a regular and fixed schedule of equal payments. Pensions and retirement benefits that go for a fixed length of time are excellent examples.

While not pleasant to think about, Accidental Death & Dismemberment Insurance is an important part of many employer’s benefits packages, particularly if the work that the employee is doing is in any way dangerous. If you become completely disabled and unable to work, the most generic plans will give you a biweekly payment equal to your former wages until you turn 65-years old. An important question is, how much is that worth?

Suppose a worker makes $46,000 per year, and is injured at age 28. Let us assume that the prevailing rate is 7%, and since he is paid biweekly we will make the prevailing rate biweekly as well. To keep the math simple, let us assume he is injured on his 28th birthday. He then has $65 - 28 = 37$ years of benefits coming to him. Next, his biweekly wage is $46,000/26 = 1769.23$. The number of payments would be $26 \times 37 = 962$. Then, because $m = 26$ we have $i = r/m = 0.07/26 = 0.00269230 \cdots$. Finally, the present value is

$$PV = c \cdot \frac{1 - (1 + i)^{-n}}{i}$$

$$= 1769.23 \cdot \frac{1 - (1 + 0.00269230 \cdots)^{-962}}{0.00269230 \cdots}$$

$$= 1769.23 \cdot \frac{0.924718 \cdots}{0.00269230 \cdots}$$

$$= 607,671.83$$

You might think that the previous box is a bit of an odd problem, but when the workman gets injured, the insurance company needs to know this value for two reasons. First, they will move that sum of money from their operating fund into their annuity fund (which is operating at 7%, we are told) to make the payments available for the worker. This way the annuity fund does not have too much or too little money in it. Second, the insurance company will send an adjustor to find out who is responsible (the technical term is “liable”) and sue that person for damages. Another important point is that we did not adjust for inflation in the previous problem. Most of these policies do not, but a few will, and we will learn how to calculate that on Page 544. Lastly, it is interesting to note that these calculations are fairly sensitive to the prevailing rate, which we will now explore.
Consider the previous example, but…

1. …with a worker who makes half as much ($23,000). [Answer: $303,836.05.]

2. …with a worker who makes twice as much ($92,000). [Answer: $1,215,344.20.]

3. …what is the relationship between the annuity deposit sum and the worker’s wages?
   [Answer: a linear relationship, because double the wages doubles the sum, and half the wages halves the sum.]

Naturally, if your answer is off in the pennies in problems like this, where the answer is in the hundreds of thousands of dollars, or even in the millions, then do not worry. An answer correct to six significant figures is excellent.

Continue with the analysis of the previous example, but…

1. …with a worker who makes $46,000, but the prevailing rate is 6%.
   [Answer: $683,186.44.]

2. …with a worker who makes $46,000, but the prevailing rate is 8%.
   [Answer: $545,068.40.]

3. …with a worker who makes $46,000, but the prevailing rate is 3.5%.
   [Answer: $953,992.03.]

4. …with a worker who makes $46,000, but the prevailing rate is 14%.
   [Answer: $326,696.35.]

5. Does the same relationship exist with the prevailing rate?
   (Hint, look at # 3 and # 4.) [Answer: No, not even close.]

In a present-value annuity problem, double the PV means double the payment, and half the PV means half the payment. This means that the payment and the PV are directly proportional. That’s also why it was justifiable to use the cost-per-thousand methodology on Page 472.

*A Pause for Reflection…*

Explain in your own words why the deposit in the previous chessboard box is larger when the interest rate is lower, and is smaller when the interest is higher.
Let us imagine that you have won the lottery. You are given the choice of $26 million dollars paid over 20 years by an annual payment of $1.3 million, or a lump-sum payment of $19 million. Which should you choose? Your friend who is an investor says that 6.5% is the going rate for secure long-term investments, and so you decide to use this as the prevailing rate.

The way to solve this problem is to calculate the present value of the 20 payments of $1.3 million. Since the payments are annual, we compound annually and then \( m = 1 \) and \( i = r/m = 6.5%/1 = 6.5\% \). We have

\[
PV = c \cdot \frac{1 - (1 + i)^{-n}}{i} = 1,300,000 \cdot \frac{1 - (1 + 0.065)^{-20}}{0.065} = 14,324,059.42
\]

Therefore the lump-sum of $19 million is a much better option in this case.

Now suppose that you take the 19 million in a lump sum, and with it you purchase a 20-year annuity contract that pays 20 equal payments each year, for 20 years. The rate is 6.5% compounded annually. How much should you expect per year?

[Answer: $1,724,371.51 per year.]

The only remaining topic that I’d like to share with you in this lesson has to do with extremely long-term loans. Realistically, these loans are something you are unlikely to encounter in your professional life. However, I think that the inquiry will make known to you the deep relationship between the PV of an annuity, and the PV of a perpetuity. Furthermore, there is a slight foreshadowing of calculus, and the inquiry will conclude with with a neat trick for checking your work in PV problems. Therefore, while most instructors would skip this last little bit of the lesson, I recommend that you read it anyway.

Suppose Percival is looking to buy a house worth $700,000, and I put 5% down (or $35,000) and take out a mortgage for the remaining $665,000. Let the rate be 6% compounded monthly. What is the monthly payment for:

- A 10-year loan? [Answer: $7382.86 per month.]
- A 15-year loan? [Answer: $5611.64 per month.]
- A 20-year loan? [Answer: $4764.26 per month.]
- A 30-year loan? [Answer: $3987.01 per month.]
- A 40-year loan? [Answer: $3658.92 per month.]
- A 999-year loan? [Answer: $3325.00 per month.]

Warning, the last answer (of the last box) will vary depending on rounding errors internal to your calculator. What I listed above was calculated using a computer-algebra package and 1000 digits of precision.
Now find the total interest paid by Percival in each of the above cases.

- For 10 years? [Answer: $220,943.20.]
- For 15 years? [Answer: $345,095.20.]
- For 20 years? [Answer: $478,422.40.]
- For 30 years? [Answer: $770,323.60.]
- For 40 years? [Answer: $1,091,281.60.]
- For 999 years? [Answer: $39,195,100.00.]

What is shocking in the previous box is how little the payment falls by going from 30 years to 40 years. It is a rather small difference, less than 10%, but the loan is asking Percival to pay for far longer. Thus, 40 year mortgages are a bit stupid; this is reflected in the total interest paid, which is far more.

To be super-clear, the 40-year loan would cost $328.09 less each month, but there are an additional ten years of payments—that’s 120 additional payments! As a result, the total amount that Percival would pay is not lower but higher. In fact, it is $320,958 higher.

Continuing with the previous box, what is also cute in our analysis of Percival’s options is how little the 40-year loan and the 999-year loan differ in the monthly payment. The monthly payment for 999-years was only $333.92 cheaper than the monthly payment for 40-years.

Moreover, while the payments are essentially the same, notice how the total interest paid by Percival in the 999-year case is vastly more than the 40-year case.

While a 999-year contract might sound absurd to an American, Balliol College of Oxford University has signed a 999-year contract with the Episcopal Diocese of Oxford for St. Cross Church, to convert it to a center for its own archives. Balliol College was founded in 1263, and so being a roughly 750-year old college, it is no more irrational for them to sign a 999-year lease than 22.43-year old person to sign a 30-year mortgage contract. After all, observe that

\[
\frac{750}{999} = 0.750750750 = \frac{225225}{300000}
\]

A second example is the St. Margaret’s Institute, which leased a building for 999 years from St. John’s College of Oxford for 400,000 pounds sterling in 2008. However, St. John’s College is not a medieval organization; it was founded in 1555. Yet a third example is St. Cross College of Oxford University, which signed a 999-year lease for a building, and this college is extremely young, being founded in 1965. All of these contracts were signed in the 2005–2010 time frame.
Now that we’ve looked at a 999-year lease let us consider what would happen if we let \( n = \infty \). After all, 999-years is
\[
999 \times 12 = 11,988 \text{ months}
\]
which is much longer than a human lifespan (at least at this point in history—who knows what the future will bring). It seems as though 999 years is a good model for infinity.

Continuing with the previous box, we first realize that an infinite-duration mortgage would have a sequence of payments of equal value going on forever. However, we have already studied a financial instrument that does this: the perpetuity! Furthermore, recalling that \( r = 0.06 \) and thus \( i = 0.005 \), a perpetuity costing \( \$ 665,000 \) would have (see Page 444) as a payment
\[
V = \frac{a}{i} \\
665,000 = \frac{a}{0.005} \\
(665,000)(0.005) = 3325.00 = a
\]

Still continuing with the previous box, notice how close the perpetuity (\( \$ 3325.00 \)) is compared to the 999-year loan (\( \$ 3325.00 \)): they are identical to the penny. Of course, some rounding error might be involved, so I decided to set Maple to use 10,000 digits of precision, and calculated
\[
665,000 \times \frac{1 - (1 + 0.005)^{-999 \times 12}}{1 + 0.005} - 665,000 \times \frac{1}{0.005} = -1.4358309283010142 \cdots \times 10^{-18}
\]
which is a difference of approximately 1 quintillionth of a dollar, or a tenth of a quadrillionth of a cent. So we see that 999-years or \( n = 11,988 \) is a really, really good approximation for infinity. We have argued for, but not quite proven, that
\[
\lim_{n \to \infty} \frac{1 - (1 + i)^{-n}}{i} = \frac{c}{i}
\]
This can be said more simply: as \( n \) goes to \( \infty \), the PV of the annuity approaches the PV of a perpetuity.

The analysis of the previous box can actually be used to check your work in any PV problem—provided the duration is long. For any somewhat long annuity, the PV of the annuity should be a bit less than the PV of a perpetuity. A good guideline for “somewhat long” is 30 or more years. Let’s see this in action:

The first example in this lesson has \( \$ 2000 \) payments, once a month, for 30 years. The \( r \) was 6%, so \( i = 0.005 \). A perpetuity-due at that value would be \( 2000/0.005 = 400,000 \). The PV was \( \$ 333,583 \), which is slightly less. Of course, for the 999-year mortgage above, the two PVs were much closer.
The solution to the chessboard box on the amortization table from Page 474 is given below:

<table>
<thead>
<tr>
<th>Period</th>
<th>Starting</th>
<th>Interest</th>
<th>Payment</th>
<th>Ending</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3000.00</td>
<td>$30.00</td>
<td>$517.64</td>
<td>$2512.36</td>
</tr>
<tr>
<td>2</td>
<td>$2512.36</td>
<td>$25.12</td>
<td>$517.64</td>
<td>$2019.84</td>
</tr>
<tr>
<td>3</td>
<td>$2019.84</td>
<td>$20.20</td>
<td>$517.64</td>
<td>$1522.40</td>
</tr>
<tr>
<td>4</td>
<td>$1522.40</td>
<td>$15.22</td>
<td>$517.64</td>
<td>$1019.98</td>
</tr>
<tr>
<td>5</td>
<td>$1019.98</td>
<td>$10.20</td>
<td>$517.64</td>
<td>$512.54</td>
</tr>
<tr>
<td>6</td>
<td>$512.54</td>
<td>$5.13</td>
<td>$517.64</td>
<td>$0.03</td>
</tr>
</tbody>
</table>

We have learned the following skills in this lesson:

- To derive the PV formula as the difference of two perpetuities.
- To derive the PV formula as a geometric series stopping early.
- To use the PV formula to calculate parameters of loans with multiple payments.
- To find the cost-per-thousand, or compounding factor, of an annuity.
- To calculate the compounding factor, which is also called the growth factor or power of compounding, and which is denoted by a special symbol.
- To construct an amortization table for an annuity.
- To measure the impact of a down-payment requirement on the maximum size of a loan.
- To calculate how long it will take to pay off an annuity-style loan, both according to schedule and with overpayments.
- To check the type of problem in the previous bullet by plugging in the two nearest values and seeing if they straddle the desired value.
- To calculate the total interest paid and the finance charge of a loan.
- To observe that the PV varies linearly with the payment, but very sensitively to $i$.
- To calculate the tradeoff between lump-sum lottery payments and annuity-style payments.
- To see how the payment varies with the duration of the loan, and to see the futility of 40-year mortgages compared with 30-year ones.
- To calculate the tradeoff between a cash rebate and 0% financing, or a cash-back incentive and 0% financing, when buying a car.
- To see that perpetuities have a PV equal to the limit of the PV of an annuity, when the number of payments goes to infinity. Furthermore, how to use that to check one’s work in problems with very long loans. (It is such an easy and effective way to check your work!)
- As well as the vocabulary terms: Accidental Death & Dismemberment Insurance, annuity, annuity forborne, compounding factor, finance charge, growth factor, power of compounding, zero-percent financing.